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# Methods for knowledge reduction in inconsistent ordered information systems

**Wei-Hua Xu · Wen-Xiu Zhang**

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**Abstract** Knowledge reduction is one of the most important problems in the study of rough set theory. However, in real-world, most of information systems are based on dominance relations in stead of the classical equivalence relation because of various factors. The ordering of properties of attributes plays a crucial role in those systems. To acquire brief decision rules from the systems, knowledge reductions are needed. The main objective of this paper is to deal with this problem. The distribution reduction and maximum distribution reduction are proposed in inconsistent ordered information systems. Moreover, properties and relationship between them are discussed. Furthermore, judgment theorem and discernibility matrix are obtained, from which an approach to knowledge reductions can be provided in inconsistent ordered information systems.

**Keywords** Rough set · Consistent set · Knowledge reduction · Discernibility matrix

**Mathematics Subject Classification (2000)** TP18 · O159

## 1 Introduction

The rough set theory, proposed by Pawlak in the early 1980s [1], is an extension of the classical set theory for modeling uncertainty or imprecision information. The research has recently roused great interest in the theoretical and application fronts, such as machine learning, pattern recognition, data analysis, and so on.

Knowledge reduction is one of the hot research topics of rough set theory. Much study on this area had been reported and many useful results were obtained [2–8,

W.-H. Xu (✉) · W.-X. Zhang
Institute for Information and System Sciences, School of Science, Xi'an Jiaotong University, Xi'an, Shaan'xi 710049, China
e-mail: datongxuweihua@126.com

19, 20]. However, most work was based on consistent information systems, and the main methodology has been developed under equivalence relations(indiscernibility relations). In practice, most of information systems are not only inconsistent, but also based on dominance relations because of various factors. The ordering of properties of attributes plays a crucial role in those systems. For this reason, Greco, Matarazzo, and Slowinski [9–13] proposed an extension rough sets theory, called the dominance-based rough sets approach (DRSA) to take into account the ordering properties of attributes. This innovation is mainly based on substitution of the indiscernibility relation by a dominance relation. In DRSA condition attributes and classes are preference ordered. And many studies have been made in DRSA [14–18]. But useful results of knowledge reductions are very poor in inconsistent ordered information systems until now.

In this paper the main aim is to study the problem. The distribution reduction and maximum distribution reduction are proposed in inconsistent ordered information systems. Moreover, properties and relationship between them are discussed. Furthermore, judgment theorem and discernibility matrix are obtained, from which an approach to knowledge reductions can be provided in inconsistent ordered information systems.

## 2 Rough sets and ordered information systems

The following recalls necessary concepts and preliminaries required in the sequel of our work. Detailed description of the theory can be found in [5, 17].

An information system with decisions is an ordered quadruple $\mathcal{I} = (U, A \cup D, F, G)$, where

$U = \{x_1, x_2, \ldots, x_n\}$ is a non-empty finite set of objects;
$A \cup D$ is a non-empty finite attributes set;
$A = \{a_1, a_2, \ldots, a_p\}$ denotes the set of condition attributes;
$D = \{d_1, d_2, \ldots, d_q\}$ denotes the set of decision attributes,
and $A \cap D = \phi$;
$F = \{f_k | U \to V_k, k \leq p\}$, $f_k(x)$ is the value of $a_k$ on $x \in U$, $V_k$ is the domain of $a_k$, $a_k \in A$;
$G = \{g_{k'} | U \to V_{k'}, k' \leq q\}$, $g_{k'}(x)$ is the value of $d_{k'}$ on $x \in U$, $V_{k'}$ is the domain of $d_{k'}$, $d_{k'} \in D$.

In an information systems, if the domain of an attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion.

**Definition 2.1** An information system is called an *ordered information system*(OIS) if all condition attributes are criterions.

Assumed that the domain of a criterion $a \in A$ is complete pre-ordered by an outranking relation $\succeq_a$, then $x \succeq_a y$ means that $x$ is at least as good as $y$ with respect to criterion $a$. And we can say that $x$ dominates $y$. In the following, without any loss of generality, we consider condition and decision criterions having a numerical domain, that is, $V_a \subseteq \mathcal{R}$($\mathcal{R}$ denotes the set of real numbers).

We define $x \succeq y$ by $f_a(x) \geq f_a(y)$ according to increasing preference, where $a \in A$ and $x, y \in U$. For a subset of attributes $B \subseteq A$, $x \succeq_B y$ means that $x \succeq_a y$ for any $a \in B$. That is to say $x$ dominates $y$ with respect to all attributes in $B$. Furthermore, we denote $x \succeq_B y$ by $x R_B^{\succeq} y$. In general, we indicate an ordered information systems with decision by $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$. Thus the following definition can be obtained.

**Definition 2.2** Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an ordered information system with decisions, for $B \subseteq A$, denote

$$R_B^{\succeq} = \big\{(x_i, x_j) \in U \times U | f_l(x_i) \geq f_l(x_j), \forall a_l \in B\big\};$$

$$R_D^{\succeq} = \big\{(x_i, x_j) \in U \times U | g_m(x_i) \geq g_m(x_j), \forall d_m \in D\big\}.$$

$R_B^{\succeq}$ and $R_D^{\succeq}$ are called *dominance relations of information system* $\mathcal{I}^{\succeq}$.

If we denote

$$\begin{aligned}[x_i]_B^{\succeq} &= \big\{x_j \in U | (x_j, x_i) \in R_B^{\succeq}\big\} \\ &= \big\{x_j \in U | f_l(x_j) \geq f_l(x_i), \forall a_l \in B\big\}; \\ [x_i]_D^{\succeq} &= \big\{x_j \in U | (x_j, x_i) \in R_D^{\succeq}\big\} \\ &= \big\{x_j \in U | g_m(x_j) \geq g_m(x_i), \forall d_m \in D\big\},\end{aligned}$$

then the following properties of a dominance relation are trivial.

**Proposition 2.1** *Let $R_A^{\succeq}$ be a dominance relation. The following hold.*

(1) $R_A^{\succeq}$ *is reflexive,transitive, but not symmetric, so it is not an equivalence relation.*
(2) *If $B \subseteq A$, then $R_A^{\succeq} \subseteq R_B^{\succeq}$.*
(3) *If $B \subseteq A$, then $[x_i]_A^{\succeq} \subseteq [x_i]_B^{\succeq}$*
(4) *If $x_j \in [x_i]_A^{\succeq}$, then $[x_j]_A^{\succeq} \subseteq [x_i]_A^{\succeq}$ and $[x_i]_A^{\succeq} = \bigcup\{[x_j]_A^{\succeq} \mid x_j \in [x_i]_A^{\succeq}\}$.*
(5) $[x_j]_A^{\succeq} = [x_i]_A^{\succeq}$ *iff* $f_a(x_i) = f_a(x_j)$ $(\forall a \in A)$.
(6) $\mathcal{J} = \bigcup\{[x]_A^{\succeq} \mid x \in U\}$ *constitute a covering of $U$.*

For any subset $X$ of $U$, and $A$ of $\mathcal{I}^{\succeq}$ define

$$\underline{R_A^{\succeq}}(X) = \big\{x \in U | [x]_A^{\succeq} \subseteq X\big\};$$

$$\overline{R_A^{\succeq}}(X) = \big\{x \in U | [x]_A^{\succeq} \cap X \neq \phi\big\},$$

$\underline{R_A^{\succeq}}(X)$ and $\overline{R_A^{\succeq}}(x)$ are said to be the lower and upper approximation of $X$ with respect to a dominance relation $R_A^{\succeq}$. And the approximations have also some properties which are similar to those of Pawlak approximation spaces.

**Proposition 2.2** *Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an information system based on dominance relation and $X, Y \subseteq U$, then its lower and upper approximations satisfy the following properties.*

**Table 1**

| $U$ | $a_1$ | $a_2$ | $a_3$ | $d$ |
|---|---|---|---|---|
| $x_1$ | 1 | 2 | 1 | 3 |
| $x_2$ | 3 | 2 | 2 | 2 |
| $x_3$ | 1 | 1 | 2 | 1 |
| $x_4$ | 2 | 1 | 3 | 2 |
| $x_5$ | 3 | 3 | 2 | 3 |
| $x_6$ | 3 | 2 | 3 | 1 |

(1) $\underline{R_A^{\succeq}}(X) \subseteq X \subseteq \overline{R_A^{\succeq}}(X)$.

(2) $\overline{R_A^{\succeq}}(X \cup Y) = \overline{R_A^{\succeq}}(X) \cup \overline{R_A^{\succeq}}(Y)$; $\quad \underline{R_A^{\succeq}}(X \cap Y) = \underline{R_A^{\succeq}}(X) \cap \underline{R_A^{\succeq}}(Y)$.

(3) $\underline{R_A^{\succeq}}(X) \cup \underline{R^{\succeq}}(Y) \subseteq \underline{R_A^{\succeq}}(X \cup Y)$; $\quad \overline{R_A^{\succeq}}(X \cap Y) \subseteq \overline{R_A^{\succeq}}(X) \cap \overline{R^{\succeq}}(Y)$.

(4) $\underline{R_X^{\succeq}}(\sim X) = \sim \overline{R_A^{\succeq}}(X)$; $\quad \overline{R_A^{\succeq}}(\sim X) = \sim \underline{R_A^{\succeq}}(X)$.

(5) $\underline{R_A^{\succeq}}(U) = U$; $\quad \overline{R_A^{\succeq}}(\phi) = \phi$.

(6) $\underline{R_A^{\succeq}}(X) \subseteq \underline{R_A^{\succeq}}(\underline{R_A^{\succeq}}(X))$; $\quad \overline{R_A^{\succeq}}(\overline{R_A^{\succeq}}(X)) \subseteq \overline{R_A^{\succeq}}(X)$.

(7) *If* $X \subseteq Y$, *then* $\underline{R_A^{\succeq}}(X) \subseteq \underline{R_A^{\succeq}}(Y)$ *and* $\overline{R_A^{\succeq}}(X) \subseteq \overline{R_A^{\succeq}}(Y)$.

*where* $\sim X$ *is the complement of* $X$.

**Definition 2.3** For an ordered information system with decisions $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$, if $R_A^{\succeq} \subseteq R_D^{\succeq}$, then this information system is *consistent*, otherwise, this information system is *inconsistent*.

*Example 2.1* An ordered information system is given in Table 1.

From the table, we have

$$[x_1]_A^{\succeq} = \{x_1, x_2, x_5, x_6\};$$
$$[x_2]_A^{\succeq} = \{x_2, x_5, x_6\};$$
$$[x_3]_A^{\succeq} = \{x_2, x_3, x_4, x_5, x_6\};$$
$$[x_4]_A^{\succeq} = \{x_4, x_6\};$$
$$[x_5]_A^{\succeq} = \{x_5\};$$
$$[x_6]_A^{\succeq} = \{x_6\};$$

and

$$[x_1]_d^{\succeq} = [x_5]_d^{\succeq} = \{x_1, x_5\};$$
$$[x_2]_d^{\succeq} = [x_4]_d^{\succeq} = \{x_1, x_2, x_4, x_5\};$$
$$[x_3]_d^{\succeq} = [x_6]_d^{\succeq} = \{x_1, x_2, x_3, x_4, x_5, x_6\}.$$

Obviously, by the above, we have $R_A^{\succeq} \not\subseteq R_d^{\succeq}$, so the system in Table 1 is inconsistent.

For simple description, the following information system with decisions are based on dominance relations, i.e., ordered information systems.

## 3 Theories of knowledge reduction in inconsistent ordered information systems

Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an information system with decisions, and $R_B^{\succeq}$, $R_D^{\succeq}$ be dominance relations derived from condition attributes set $A$ and decision attributes set $D$ respectively. For $B \subseteq A$, denote

$$U/R_B^{\succeq} = \left\{[x_i]_B^{\succeq} \mid x_i \in U\right\};$$
$$U/R_d^{\succeq} = \{D_1, D_2, \ldots, D_r\};$$
$$\mu_B^{\succeq}(x) = \left(\frac{|D_1 \cap [x]_B^{\succeq}|}{|U|}, \frac{|D_2 \cap [x]_B^{\succeq}|}{|U|}, \ldots, \frac{|D_r \cap [x]_B^{\succeq}|}{|U|}\right);$$
$$\gamma_B^{\succeq}(x) = \max\left\{\frac{|D_1 \cap [x]_B^{\succeq}|}{|U|}, \frac{|D_2 \cap [x]_B^{\succeq}|}{|U|}, \ldots, \frac{|D_r \cap [x]_B^{\succeq}|}{|U|}\right\},$$

where $[x]_B^{\succeq} = \{y \in U | (x, y) \in R_B^{\succeq}\}$. Furthermore, we said $\mu_B^{\succeq}(x)$ be distribution function about attributions sets $B$, and $\gamma_B^{\succeq}(x)$ be maximum distribution function about attributions sets $B$.

**Definition 3.1** Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$ be two vectors with $n$ dimensions. If $a_i = b_i$ $(i = 1, 2, \ldots, n)$, we say that $\alpha$ *is equal to* $\beta$, denoted by $\alpha = \beta$. If $a_i \leq b_i$ $(i = 1, 2, \ldots, n)$, we say that $\alpha$ *is less than* $\beta$, denoted by $\alpha \leq \beta$. Otherwise, If it exists $i_0$ $(i_0 \in \{1, 2, \ldots, n\})$ such that $a_{i_0} > b_{i_0}$, we say $\alpha$ *is not less than* $\beta$, denoted by $\alpha \not\leq \beta$.

Such as $(1, 2, 3) \not\leq (1, 1, 4)$, $(1, 1, 4) \not\leq (1, 2, 3)$. From the above, we can have the following propositions immediately.

**Proposition 3.1**

(1) *If* $B \subseteq A$, *then* $\mu_A^{\succeq}(x) \leq \mu_B^{\succeq}(x), \forall x \in U$.
(2) *If* $B \subseteq A$, *then* $\gamma_A^{\succeq}(x) \leq \gamma_B^{\succeq}(x), \forall x \in U$.
(3) *If* $[y]_B^{\succeq} \supseteq [x]_B^{\succeq}$, *then* $\mu_B^{\succeq}(y) \leq \mu_B^{\succeq}(x), \forall x, y \in U$.
(4) *If* $[y]_B^{\succeq} \supseteq [x]_B^{\succeq}$, *then* $\gamma_B^{\succeq}(y) \leq \gamma_B^{\succeq}(x), \forall x, y \in U$.

**Definition 3.2** Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an Inconsistent information system. If $\mu_B^{\succeq}(x) = \mu_A^{\succeq}(x)$, for all $x \in U$, we say that $B$ is a *distribution consistent set* of $\mathcal{I}^{\succeq}$. If $B$ is a distribution consistent set, and no proper subset of $B$ is distribution consistent set, then $B$ is called a *distribution consistent reduction* of $\mathcal{I}^{\succeq}$.

**Definition 3.3** Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an inconsistent information system. If $\gamma_B^{\succeq}(x) = \gamma_A^{\succeq}(x)$, for all $x \in U$, we say that $B$ is a *maximum distribution consistent set* of $\mathcal{I}^{\succeq}$. If $B$ is a maximum distribution set, and no proper subset of $B$ is maximum distribution consistent set, then $B$ is called a *maximum distribution consistent reduction* of $\mathcal{I}^{\succeq}$.

*Example 3.1* Consider the system in Table 1. For the system in Table 1, we denote

$$D_1 = [x_1]_d^{\succeq} = [x_5]_d^{\succeq}, \qquad D_2 = [x_2]_d^{\succeq} = [x_4]_d^{\succeq}, \qquad D_3 = [x_3]_d^{\succeq} = [x_6]_d^{\succeq}.$$

And we can have

$$\mu_A^{\succeq}(x_1) = \left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right); \qquad \mu_A^{\succeq}(x_2) = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right);$$
$$\mu_A^{\succeq}(x_3) = \left(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}\right); \qquad \mu_A^{\succeq}(x_4) = \left(0, \frac{1}{6}, \frac{1}{3}\right);$$
$$\mu_A^{\succeq}(x_5) = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right); \qquad \mu_A^{\succeq}(x_6) = \left(0, 0, \frac{1}{6}\right);$$
$$\gamma_A^{\succeq}(x_1) = \frac{2}{3}; \qquad \gamma_A^{\succeq}(x_2) = \frac{1}{2}; \qquad \gamma_A^{\succeq}(x_3) = \frac{5}{6};$$
$$\gamma_A^{\succeq}(x_4) = \frac{1}{3}; \qquad \gamma_A^{\succeq}(x_5) = \frac{1}{6}; \qquad \gamma_A^{\succeq}(x_6) = \frac{1}{6}.$$

When $B = \{a_2, a_3\}$, it can be easily checked that $[x]_A^{\succeq} = [x]_B^{\succeq}$, for all $x \in U$. So that $\mu_B^{\succeq}(x) = \mu_A^{\succeq}(x)$ and $\gamma_B^{\succeq}(x) = \gamma_A^{\succeq}(x)$ are true, and $B = \{a_2, a_3\}$ is a distribution consistent set of $\mathcal{I}^{\succeq}$. Furthermore, we can examine that $\{a_2\}$ and $\{a_3\}$ are not consistent set of $\mathcal{I}^{\succeq}$. That is to say $B = \{a_2, a_3\}$ is a distribution reduction and maximum distribution reduction of $\mathcal{I}^{\succeq}$.

Moreover, it can be easily calculated that $B' = \{a_1, a_3\}$ and $B'' = \{a_1, a_2\}$ are not distribution consistent sets of $\mathcal{I}^{\succeq}$. Thus there exist only one distribution reduction and maximum distribution reduction of $\mathcal{I}^{\succeq}$ in the system of Table 1, which are $\{a_2, a_3\}$.

In the next, detailed judgment theorems of distribution reduction and maximum distribution reduction are obtained.

**Theorem 3.1** *Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an information system with decision, and $B \subseteq A$ is a distribution consistent set of $\mathcal{I}^{\succeq}$, then $B$ is a maximum distribution consistent set of $\mathcal{I}^{\succeq}$.*

*Proof* It can be proved by definitions immediately. □

**Corollary 3.1** *Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an information system with decision, and $B \subseteq A$ is a distribution reduction of $\mathcal{I}^{\succeq}$, then $B$ is a maximum distribution reduction set of $\mathcal{I}^{\succeq}$.*

**Theorem 3.2** *Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an information system with decisions, then $B \subseteq A$ is a distribution consistent set of $\mathcal{I}^{\succeq}$ if and only if when $\mu_A^{\succeq}(y) \not\leq \mu_A^{\succeq}(x)$, $[y]_B^{\succeq} \not\subseteq [x]_B^{\succeq}$ always holds for any $x, y \in U$.*

*Proof* Assume that when $\mu_A^{\succeq}(y) \not\leq \mu_A^{\succeq}(x)$, $[y]_B^{\succeq} \not\subseteq [x]_B^{\succeq}$ doesn't hold, that implies $[y]_B^{\succeq} \subseteq [x]_B^{\succeq}$. So we can obtain $\mu_B^{\succeq}(y) \leq \mu_B^{\succeq}(x)$ by Proposition 3.1(3). On the other hand, since $B$ is a distribution consistent set of $\mathcal{I}^{\succeq}$, we have $\mu_A^{\succeq}(x) = \mu_B^{\succeq}(x)$ and $\mu_A^{\succeq}(y) = \mu_B^{\succeq}(y)$. Hence we can get $\mu_B^{\succeq} \leq \mu_B^{\succeq}(x)$, which is a contradiction.

Conversely, we only prove $\mu_B^{\succeq}(x) \leq \mu_A^{\succeq}(x)$ by Proposition 2.1(2).

Obviously it is true when $\mu_B^{\succeq}(x) = 0$. In the next, we prove the case of $\mu_B^{\succeq}(x) \neq 0$. For all $x, y \in U$, if $\mu_A^{\succeq}(y) \not\leq \mu_A^{\succeq}(x)$ implies $[y]_B^{\succeq} \not\subseteq [x]_B^{\succeq}$, then which means that $[y]_B^{\succeq} \subseteq [x]_B^{\succeq}$ implies $\mu_A^{\succeq}(y) \leq \mu_A^{\succeq}(x)$.

On the other hand, when $\frac{|D_i \cap [x]_B^{\succeq}|}{|U|} \neq 0$, we can have $|D_i \cap [x]_B^{\succeq}| \neq 0$ $(i = 1, 2, \ldots)$. So we assume that $y_i \in [x]_B^{\succeq} \cap D_i$, then $y \in [x]_B^{\succeq}$ and $y \in D_i$. By Proposition 2.1(4), we obtain that $[y_i]_B^{\succeq} \subseteq [x]_B^{\succeq}$ is true, which implies $\mu_A^{\succeq}(y_i) \leq \mu_A^{\succeq}(x)$. Since $y_i \in [y_i]_A^{\succeq}$, we have $y_i \in D_i \cap [y_i]_A^{\succeq}$, and that is $|D_i \cap [x]_A^{\succeq}| \leq |D_i \cap [y_i]_A^{\succeq}|$. So we get $\mu_B^{\succeq}(x) \leq \mu_A^{\succeq}(y_i)$. Hence, we have $\mu_B^{\succeq}(x) \leq \mu_A^{\succeq}(x)$, i.e., $B$ is a distribution consistent set of $\mathcal{I}^{\succeq}$. □

**Corollary 3.2** *Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an information system with decisions, then $B \subseteq A$ is a maximum distribution consistent set of $\mathcal{I}^{\succeq}$ if and only if when $\gamma_A^{\succeq}(y) > \mu_A^{\succeq}(x)$, $[y]_B^{\succeq} \not\subseteq [x]_B^{\succeq}$ always holds for any $x, y \in U$.*

## 4 Methods for knowledge reduction in inconsistent ordered information systems

This section provides approaches to distribution reduction in inconsistent information systems. Let first give the following notions.

**Definition 4.1** Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an information system. We denote

$$D^*_{\mu^{\succeq}} = \left\{(x_i, x_j) : \mu_A^{\succeq}(x_i) \not\leq \mu_A^{\succeq}(x_j)\right\},$$

$$D^*_{\gamma^{\succeq}} = \left\{(x_i, x_j) : \gamma_A^{\succeq}(x_i) > \gamma_A^{\succeq}(x_j)\right\}.$$

Denoted by $f_{a_k}$ the value of $a_k$ w.r.t. the object $x$. Define

$$D_{\mu^{\succeq}}(x_i, x_j) = \begin{cases} \{a_k \in A : f_{a_k}(x_i) > f_{a_k}(x_j)\}, & (x_i, x_j) \in D^*_{\mu^{\succeq}}, \\ \phi, & (x_i, x_j) \notin D^*_{\mu^{\succeq}}, \end{cases}$$

$$D_{\gamma^{\succeq}}(x_i, x_j) = \begin{cases} \{a_k \in A : f_{a_k}(x_i) > f_{a_k}(x_j)\}, & (x_i, x_j) \in D^*_{\gamma^{\succeq}}, \\ \phi, & (x_i, x_j) \notin D^*_{\gamma^{\succeq}}. \end{cases}$$

Then $D_{\mu^{\succeq}}(x_i, x_j)$ and $D_{\gamma^{\succeq}}(x_i, x_j)$ are said to be *distribution discernibility attributes set* and *maximum distribution discernibility attributes set*, respectively. And matrices $M_{\mu^{\succeq}} = (D_{\mu^{\succeq}}(x_i, x_j), x_i, x_j \in U)$ and $M_{\gamma^{\succeq}} = (D_{\gamma^{\succeq}}(x_i, x_j), x_i, x_j \in U)$ are referred as to distribution discernibility matrix and maximum distribution discernibility matrix of $\mathcal{I}^{\succeq}$ respectively.

**Theorem 4.1** *Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an information system, $B \subseteq A$, then $B$ is a distribution consistent set if and only if $B \cap D_{\mu^{\succeq}}(x, y) \neq \phi$, for all $(x, y) \in D^*_{\mu^{\succeq}}$.*

*Proof* Assume that $B$ is a distribution consistent set of $\mathcal{I}^{\succeq}$. For any $(x, y) \in D^*_{\mu^{\succeq}}$, we can obtain $\mu^{\succeq}_A(x) \npreceq \mu^{\succeq}_A(y)$. From the Theorem 3.2, we have $[y]^{\succeq}_B \nsubseteq [x]^{\succeq}_B$. Thus means there exist the following three cases between $[x]^{\succeq}_B$ and $[y]^{\succeq}_B$, which are (1) $[x]^{\succeq}_B \subset [y]^{\succeq}_B$, (2) $[x]^{\succeq}_B \cap [y]^{\succeq}_B = \phi$, (3) both $[x]^{\succeq}_B \cap [y]^{\succeq}_B \subset [x]^{\succeq}_B$ and $[x]^{\succeq}_B \cap [y]^{\succeq}_B \subset [y]^{\succeq}_B$. We will prove that $B \cap D(_{\mu^{\succeq}}x, y) \neq \phi$ always holds in every case.

*Case* 1. If $[x]^{\succeq}_B \subset [y]^{\succeq}_B$, then there necessarily exist an element $z \in [y]^{\succeq}_B$, but $z \notin [x]^{\succeq}_B$. From $z \notin [x]^{\succeq}_B$, we can certainly find an element $a_k \in B$, such that $f_{a_k}(x) > f_{a_k}(z)$. On the other hand, the fact $f_{a_k}(y) \geq f_{a_k}(z)$ is true according to $z \in [y]^{\succeq}_B$. From the above, we can obtain $f_{a_k}(x) > f_{a_k}(y)$. Hence, we have $a_k \in D_{\mu^{\succeq}}(x, y)$, i.e., $B \cap D_{\mu^{\succeq}}(x, y) \neq \phi$.

*Case* 2. If $[x]^{\succeq}_B \cap [y]^{\succeq}_B = \phi$, then there exists necessarily an element $a_k \in B$, such that $f_{a_k}(x) > f_{a_k}(y)$, i.e. $B \cap D_{\mu^{\succeq}}(x, y) \neq \phi$. Otherwise, if for all $a_l \in B$, $f_{a_l}(x) \geq f_{a_l}(y)$ always holds, then we observe $y \in [x]^{\succeq}_B$. This is contradiction.

*Case* 3. The proof is similar to Case 1, because we can also find certainly an element $z \in [y]^{\succeq}_B$, but $z \notin [x]^{\succeq}_B$ in the case.

Thus we can conclude that $B \cap D_{\mu^{\succeq}}(x, y) \neq \phi$ for all $(x, y) \in D^*_{\mu^{\succeq}}$.

Hence, if $B$ is a distribution consistent set, then $B \cap D_{\mu^{\succeq}}(x, y) \neq \phi$, for all $(x, y) \in D^*_{\mu^{\succeq}}$.

Conversely, if every $(x, y) \in D^*_{\mu^{\succeq}}$ satisfies $B \cap D_{\mu^{\succeq}}(x, y) \neq \phi$, then we can select an $a_k \in B$, such that $a_k \in D_{\mu^{\succeq}}(x, y)$. That is $f_{a_k}(x) > f_{a_k}(y)$, so $y \notin [x]^{\succeq}_B$. Since $y \in [y]^{\succeq}_B$ is true, we can obtain $[x]^{\succeq}_B \cap [y]^{\succeq}_B \neq [y]^{\succeq}_B$, that is $[y]^{\succeq}_B \nsubseteq [x]^{\succeq}_B$. On the other hand, since $(x, y) \in D^*_{\mu^{\succeq}}$, we have $\mu^{\succeq}_A(x) \npreceq \mu^{\succeq}_A(y)$. Hence, we find that when $\mu^{\succeq}_A(x) \npreceq \mu^{\succeq}_A(y)$, $[y]^{\succeq}_B \nsubseteq [x]^{\succeq}_B$ holds. Thus we get that $B$ is a distribution consistent set of $\mathcal{I}^{\succeq}$ in term of Theorem 3.2. □

**Theorem 4.2** *Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an information system, $B \subseteq A$, then $B$ is a maximum distribution consistent set if and only if $B \cap D_{\gamma^{\succeq}}(x, y) \neq \phi$, for all $(x, y) \in D^*_{\gamma^{\succeq}}$.*

*Proof* It is similar to the above theorem. □

**Definition 4.1** Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an information system, and $M_{\mu^{\succeq}}$ and $M_{\gamma^{\succeq}}$ be distribution discernibility matrix and maximum distribution discernibility matrix of $\mathcal{I}^{\succeq}$ respectively. Denote

$$F_{\mu^{\succeq}} = \wedge\Big\{ \vee \{a_k : a_k \in D_{\mu^{\succeq}}(x_i, x_j)\}, x_i, x_j \in U\Big\}$$

$$= \wedge\Big\{ \vee \{a_k : a_k \in D_{\mu^{\succeq}}(x_i, x_j)\}, x_i, x_j \in D^*_{\mu^{\succeq}}\Big\};$$

$$F_{\gamma^{\succeq}} = \wedge\Big\{ \vee \{a_k : a_k \in D_{\gamma^{\succeq}}(x_i, x_j)\}, x_i, x_j \in U\Big\}$$

$$= \wedge\Big\{ \vee \{a_k : a_k \in D_{\gamma^{\succeq}}(x_i, x_j)\Big\}, x_i, x_j \in D^*_{\gamma^{\succeq}}\Big\},$$

$F_{\mu^{\succeq}}$ and $F_{\gamma^{\succeq}}$ are called *discernibility formula of distribution discernibility* and *maximum distribution*, respectively.

**Theorem 4.3** *Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an information system. The minimal disjunctive normal form of discernibility formula of distribution is*

$$F_{\mu^{\succeq}} = \bigvee_{k=1}^{p} \left( \bigwedge_{s=1}^{q_k} \right) a_s.$$

*Denote $B^k_{\mu^{\succeq}} = \{a_s : s = 1, 2, \ldots, q_k\}$, then $\{B^k_{\mu^{\succeq}} : k = 1, 2, \ldots, p\}$ is just set of all distribution reductions of $\mathcal{I}^{\succeq}$.*

*Proof* It follows directly from Theorem 4.1 and the definition of minimal disjunctive normal of the discernibility formula of distribution. □

**Theorem 4.4** *Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an information system. The minimal disjunctive normal form of discernibility formula of maximum distribution is*

$$F_{\gamma^{\succeq}} = \bigvee_{k=1}^{p} \left( \bigwedge_{s=1}^{q_k} \right) a'_s.$$

*Denote $B^k_{\gamma^{\succeq}} = \{a'_s : s = 1, 2, \ldots, q_k\}$, then $\{B^k_{\gamma^{\succeq}} : k = 1, 2, \ldots, p\}$ is just set of all maximum distribution reductions of $\mathcal{I}^{\succeq}$.*

*Proof* It is similar to the above theorem. □

Theorems 4.3 and 4.4 provides a practical approach to distribution and maximum distribution reductions of information systems with decisions based on dominance relation. The following we will consider the system in Table 1 using this approach.

*Example 4.1* For the system in Table 1, the function of distribution and maximum distribution have been obtained in Example 3.1. In additional, we can have

$$D^*_{\mu^{\succeq}} = \{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_1, x_5), (x_1, x_6), (x_2, x_4), (x_2, x_5), (x_2, x_6), (x_3, x_1), (x_3, x_2), (x_3, x_4), (x_3, x_5), (x_3, x_6), (x_4, x_5), (x_4, x_6), (x_5, x_4), (x_5, x_6)\};$$

$$D^*_{\gamma^{\succeq}} = \{(x_1, x_2), (x_1, x_4), (x_1, x_5), (x_1, x_6), (x_2, x_4), (x_2, x_5), (x_2, x_6), (x_3, x_1), (x_3, x_2), (x_3, x_4), (x_3, x_5), (x_3, x_6), (x_4, x_5), (x_4, x_6)\};$$

**Table 2** $(M_{\mu^{\succeq}})$

| $M_{\mu^{\succeq}}$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| $x_1$ | $\phi$ | $\phi$ | $a_2$ | $a_2$ | $\phi$ | $\phi$ |
| $x_2$ | $\phi$ | $\phi$ | $\phi$ | $a_1, a_2$ | $\phi$ | $\phi$ |
| $x_3$ | $a_3$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ |
| $x_4$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ | $a_3$ | $\phi$ |
| $x_5$ | $\phi$ | $\phi$ | $\phi$ | $a_1, a_2$ | $\phi$ | $a_2$ |
| $x_6$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ |

**Table 3** $(M_{\gamma^{\succeq}})$

| $M_{\gamma^{\succeq}}$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| $x_1$ | $\phi$ | $\phi$ | $\phi$ | $a_2$ | $\phi$ | $\phi$ |
| $x_2$ | $\phi$ | $\phi$ | $\phi$ | $a_1, a_2$ | $\phi$ | $\phi$ |
| $x_3$ | $a_3$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ |
| $x_4$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ | $a_3$ | $\phi$ |
| $x_5$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ |
| $x_6$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ |

The following tables (Tables 2, 3) are the distribution and maximum distribution discernibility matrices of system in Table 1.

Consequently, we have

$$F_{\mu^{\succeq}} = F_{\gamma^{\succeq}} = a_2 \wedge a_3 \wedge (a_1 \vee a_2) = a_2 \wedge a_3$$

Therefore, we obtain that $\{a_2, a_3\}$ is all distribution reductions of information system in Table 1, and is also all maximum distribution reduction which accords with the result of Example 3.1.

## 5 Conclusion

It is well known that most of information systems are not only inconsistent, but also based on dominance relations because of various factors in practice. Therefore, it is important to study the knowledge reductions in inconsistent formation systems. In this paper, we are concerned with approaches to the problem. The distribution reduction and maximum distribution reduction are introduced in inconsistent systems, and relationship between them are examined. The judgment theorem and discernibility matrix are obtained, from which we can provide the approach to knowledge reductions in inconsistent systems based on dominance.

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**Wei-Hua Xu** received the B.S. in Science from Yanbei Normal Institute in 2001 and M.S. degrees in Science from Guangxi University in 2004, respectively. He has been studying in Institute for Information and System Science of Xi'an Jiaotong University. Main research field is fuzzy set, rough set and the knowledge discovery in artificial intelligent.

**Wen-Xiu Zhang** graduated in information theory, probability and statistics form Nankai University, China, in 1967. He currently serves as professor and doctoral supervisor at Xi'an Jiaotong University, China. He is the author and coauthor of more than 120 academic journal papers and 16 textbooks and research monographs. His research specialization covers fuzzy sets, random sets, rough sets, genetic algorithms and the mathematical foundation of information science.